Revision 3 (Solutions)

Year 11 Examination

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section One:
Calculator-free

Student Number: In figures

In words

Teacher name

Time allowed for this section

Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **35% (46 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

(a) Determine the value(s) of a for which the matrix $\begin{bmatrix} a & a \\ 3 & 2a \end{bmatrix}$ is singular. (2 marks)

Solution
Singular ⇒ determinant is zero, so require $2a^2 - 3a = 0$. $a(2a-3) = 0 \Rightarrow a = 0 \text{ or } a = \frac{3}{2}$
Specific behaviours
✓ writes determinant in terms of a and equates to 0   ✓ solves equation for a

(b) The non-singular matrix B is such that $[-3 \quad 2] \times B = [8 \quad 3]$ and $[2 \quad 6] \times B = [10 \quad 4]$.

(i) Use these results to show that $[-1 \quad 8] \times B = [18 \quad 7]$. (2 marks)

Solution
$[-3 \quad 2] \times B + [2 \quad 6] \times B = [8 \quad 3] + [10 \quad 4]$ $([-3 \quad 2] + [2 \quad 6]) \times B = [18 \quad 7]$ $[-1 \quad 8] \times B = [18 \quad 7]$
Specific behaviours
✓ uses sum of equations   ✓ illustrates distributive law

(ii) Determine $[2 \quad 1] \times B^{-1}$. (3 marks)

Solution
$([2 \quad 6] - [-3 \quad 2]) \times B = [10 \quad 4] - [8 \quad 3]$ $[5 \quad 4] \times B = [2 \quad 1]$ $[5 \quad 4] \times B \times B^{-1} = [2 \quad 1] \times B^{-1}$ $[5 \quad 4] = [2 \quad 1] \times B^{-1}$
Specific behaviours
✓ uses difference of equations   ✓ shows post-multiplication by inverse   ✓ clearly shows result

Question 2 **(6 marks)**

(a) A set of real numbers is given by $\{\sqrt{2},\ 3.\overline{14},\ \pi,\ \sqrt[3]{14}\}$. Clearly show that one of the numbers in the set is rational. (3 marks)

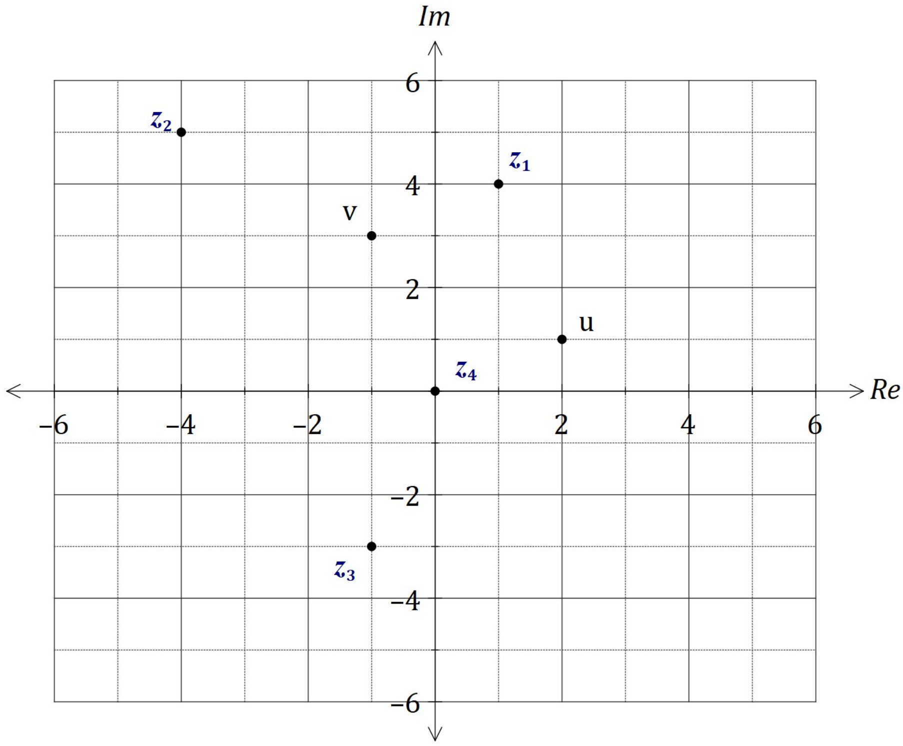
Solution
let $x = 3.\overline{14}$ Then $100x - x = 314.\overline{14} - 3.\overline{14}$ $99x = 311$ $x = \dfrac{311}{99}$ and hence is rational
Specific behaviours
✓ chooses rational number ✓ indicates use of $100x - x$ ✓ writes as rational

(b) Show that if n is one more than a multiple of three, then n^2 will also be one more than a multiple of three, where $n \in \mathbb{Z}$. (3 marks)

Solution
Let $n = 3k + 1, k \in \mathbb{Z}$ Then $n^2 = 9k^2 + 6k + 1$ $= 3(3k^2 + 2k) + 1$ Hence true
Specific behaviours
✓ writes n in required form ✓ squares n ✓ writes n^2 in required form

Question 3 **(4 marks)**

The complex numbers u and v are shown in the complex plane below.



Plot and label the following complex numbers:

(a) $z_1 = u + v$. (1 mark)

(b) $z_2 = 2v - u$. (1 mark)

(c) $z_3 = \bar{v}$. (1 mark)

(d) $z_4 = \overline{u + v} - \bar{u} - \bar{v}$. (1 mark)

Solution
See graph
Specific behaviours
✓ z_1 ✓ z_2 ✓ z_3 ✓ z_4

Question 4 **(8 marks)**

(a) Evaluate $\frac{3!7!}{9!}$. (1 mark)

$$x = \frac{3\times 2\times 7!}{9\times 8\times 7!} = \frac{1}{12}$$

(b) Determine the number of different permutations of the letters in the word NEEDLED. (2 marks)

$$n = \frac{7!}{3!2!}$$

$$= \frac{7\times 6\times 5\times 4}{2} = 420$$

(c) A password is formed using all seven of the characters \$, %, @, Y, Z, 8 and 9 just once. Determine the number of different passwords that are possible in which all the symbols are adjacent, all the letters are adjacent and all the digits are adjacent. (3 marks)

CLD: $3!\times 2!\times 2! = 24$

CLD arranged $3!$ ways

$24\times 3! = 144$

(d) Determine the least number of randomly chosen integers between 10 and 99 required to be certain that the difference of the digits in at least two of the integers is the same. (*For example, the difference of the digits in the integer* 49 *is* $9-4=5$). (2 marks)

There are 10 possible differences (from 0 to 9), which give us 10 pigeonholes to fill.

If more than 10 numbers are chosen, then at least one pigeonhole must contain two or more numbers. So at least 11 numbers must be chosen.

Question 5 **(7 marks)**

(a) Determine the value(s) of a for which the matrix $\begin{bmatrix} a & a \\ 3 & 2a \end{bmatrix}$ is singular. (2 marks)

Solution
Singular ⇒ determinant is zero, so require $2a^2 - 3a = 0$.
$a(2a-3) = 0 \Rightarrow a = 0 \text{ or } a = \frac{3}{2}$
Specific behaviours
✓ writes determinant in terms of a and equates to 0
✓ solves equation for a

(b) The non-singular matrix B is such that $[-3 \quad 2] \times B = [8 \quad 3]$ and $[2 \quad 6] \times B = [10 \quad 4]$.

(i) Use these results to show that $[-1 \quad 8] \times B = [18 \quad 7]$. (2 marks)

Solution
$[-3 \quad 2] \times B + [2 \quad 6] \times B = [8 \quad 3] + [10 \quad 4]$ $([-3 \quad 2] + [2 \quad 6]) \times B = [18 \quad 7]$ $[-1 \quad 8] \times B = [18 \quad 7]$
Specific behaviours
✓ uses sum of equations

(ii) Determine $[2 \quad 1] \times B^{-1}$. (3 marks)

Solution
$([2 \quad 6] - [-3 \quad 2]) \times B = [10 \quad 4] - [8 \quad 3]$ $[5 \quad 4] \times B = [2 \quad 1]$ $[5 \quad 4] \times B \times B^{-1} = [2 \quad 1] \times B^{-1}$ $[5 \quad 4] = [2 \quad 1] \times B^{-1}$
Specific behaviours
✓ uses difference of equations
✓ shows post-multiplication by inverse

Question 6 **(8 marks)**

(a) Determine the acute angle θ in each of the following cases:

(i) $\cos\theta = \sin 38°$. (2 marks)

Solution
$\theta = 90 - 38$ $\theta = 52°$
Specific behaviours
✓ uses complement   ✓ states angle

(ii) $\sec\theta = \operatorname{cosec} 100°$. (2 marks)

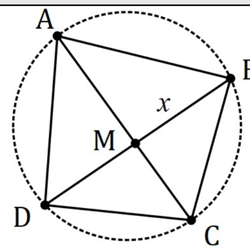
Solution
$\dfrac{1}{\cos\theta} = \dfrac{1}{\sin 100} = \dfrac{1}{\sin 80}$ $\theta = 90 - 80 = 10°$
Specific behaviours
✓ sin in first quadrant   ✓ states angle

(b) Prove that $\tan x + \sec x = \dfrac{\cos x}{1 - \sin x}$. (4 marks)

Solution
$\begin{aligned} \text{LHS} &= \frac{\sin x}{\cos x} + \frac{1}{\cos x} \\ &= \frac{(\sin x + 1)}{\cos x} \times \frac{\cos x}{\cos x} \\ &= \frac{\cos x\,(1 + \sin x)}{1 - \sin^2 x} \\ &= \frac{\cos x\,(1 + \sin x)}{(1 - \sin x)(1 + \sin x)} \\ &= \frac{\cos x}{1 - \sin x} \end{aligned}$
Specific behaviours
✓ writes LHS as single fraction   ✓ multiplies by $\cos x \div \cos x$ ✓ uses Pythagorean identity   ✓ factorises denominator and simplifies

Question 7 **(5 marks)**

Cyclic quadrilateral $ABCD$ has diagonals AC and BD that intersect at M. Given that $AM = 6$ cm, $CM = 8$ cm and $BD = 16$ cm, determine the smallest possible length of BM.

Solution
 Let $BM = x$, then $DM = 16 - x$ $x(16 - x) = 6 \times 8 = 48$ $x^2 - 16x - 48 = 0$ $(x - 8)^2 = 16$ $x = 8 \pm 4 \Rightarrow BM = 4$ cm
Specific behaviours
✓ annotated diagram ✓ expressions for BM and DM ✓ uses intersecting chord theorem to form equation ✓ factors equation ✓ states required solution